ON THE PRODUCT PROPERTY FOR THE LEMPERT FUNCTION

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ABSTRACT. We study the problem of the product property for the Lempert function with many poles and consider some properties of this function mostly for plane domains.

1. INTRODUCTION

Let A and B be at most countable non-empty subsets of domains D and G in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. We say that the Lempert function $l_{D\times G}(A\times B,\cdot)$ with pole set $A\times B$ *has the product property at the point* $(z,w)\in D\times G$ if

$$l_{D\times G}(A\times B,(z,w))=\max\{l_D(A,z),l_G(B,w)\}.$$

It is easy to see that this property is true if A and B are singletons (cf. [4]). Moreover, a necessary and sufficient condition for the product property for the Lempert function has been given in [2] (Theorem 4.1), when B is a fixed singleton and A varies over all finite subsets of D; namely, the product property holds if and only if $l_G(B,w)$ is equal to the pluricomplex Green function $g_G(B,w)$ with pole at B. Unfortunately, the proof of this result contains a gap (more precisely, there is a gap in the proof of Lemma 2.3). A main purpose of this paper is to prove a more general version of this lemma (Lemma 4 below), which allows us not only to give a corrected proof of the above mentioned result but also to refine it (Theorem 5).

Concerning the case when the pole sets are not singletons, it has been shown in [2] that the product property for the Lempert function is not true even in the case of the unit bidisc $\mathbb{D}^2$. So it is natural to study when this property holds for $\mathbb{D}^2$ when each of the pole sets A and B has two elements. A second purpose of the paper is to show that if, in addition, $l_{\mathbb{D}}(A,0)=l_{\mathbb{D}}(B,0)>0$, then the product property for $l_{\mathbb{D}^2}(A\times B,(0,0))$ is true if and only if there is a rotation sending A to B (Theorem 7). This result allows us to construct easy various examples

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of arbitrarily large pole sets of the unit disc for which the product property for the Lempert function of the bidisc is not satisfied.

The paper is organized as follows. In Section 2 we give basic facts about the Lempert function and its variations. In Section 3 we obtain explicit formulas for these functions in the plane case and descriptions of their extremal discs (which may be considered as an analogue of geodesic curves). These results are used in Section 5 to construct various counterexamples to the product property of the Lempert function. Section 4 contains proofs of Lemma 4 and Theorem 5 mentioned above.

2. Preliminaries

Let D be a domain in $\mathbb{C}^n$. Let $z \in D$ and let A be at most countable non-empty subset of D (in the paper we consider only such sets). Denote by $\mathbb{D}$ the unit disc in $\mathbb{C}$ and define

$$l_D(A,z) := \inf\{\prod_{a\in\mathbb{D}} |\lambda_a|\},$$

where the infimum is taken over all subsets $(\lambda_a)_{a\in A}$ of $\mathbb{D}$ for which there exists a $\varphi \in \mathcal{O}(\mathbb{D}, D)$ with $\varphi(0) = z$ and $\varphi(\lambda_a) = a$ for any $a \in A$ (it is shown in [6] that there are such subsets). The function $l_D(A,\cdot)$ is called the *Lempert function with poles at* A (cf. [2, 3, 5, 6, 8, 9]). Note that $l_D(a,\cdot) := l_D(\{a\},\cdot)$ is the classical Lempert function. The Lempert function is monotone under inclusion of pole sets; moreover (see [6]),

$$l_D(A,z) = \inf\{l_D(B,z) : B \text{ is a finite non-empty subset of } A\},$$

and therefore,

$$l_D(A,z) = \inf\{l_D(B,z) : \emptyset \neq B \subset A\}.$$

For any fixed $N \in \mathbb{N}^* := \mathbb{N} \cup \{\infty\}$ and $a, z \in D$, set (see [2])

$$l_D^N(a,z) := \inf\{\prod_{j=1}^{N} |\lambda_j|\},$$

where the infimum is taken over all subsets $(\lambda_j)_{j=1}^N$ of $\mathbb{D}$ for which there exists a $\varphi \in \mathcal{O}(\mathbb{D}, D)$ with $\varphi(0) = z$ and $\varphi(\lambda_j) = a$, $j = 1, \dots, N$ (obviously, there are such subsets). Note that we may also define another function, denote it by $\hat{l}_D^N(a,z)$ in a similar way as above but we allow some of the λ_j's to be equal and we count them not more than the multiplicity $\mathrm{ord}_{\lambda_j}\varphi$ of φ at λ_j. We shall show that both functions coincide.

Claim. $\hat{l}_D^N(a,z) = l_D^N(a,z)$.

Proof. We follow the ideas of A. Edigarian (see e.g. [3]) who shows the result for $N = \infty$. Assume that $N < \infty$.

It is sufficient to get that $\hat{l}_D^N(a,z) \geq l_D^N(a,z)$. Let $\epsilon > 0$ be arbitrary. Then we may find $\varphi \in \mathcal{O}(\mathbb{D}, D)$ such that $\varphi(0) = z$ and $\varphi(\xi) - a = \prod_{j=1}^l (\xi - \lambda_j)^{k_j} \psi(\xi)$, where $\lambda_1, \dots, \lambda_l$ are pairwise distinct numbers with $\sum_{j=1}^l k_j = N$, $k_j \geq 1$, and

$$\prod_{j=1}^{l} |\lambda_j^{k_j}| \leq \hat{l}_D^N(a,z) + \epsilon.$$

Consider the mapping $\varphi_t(z) = \varphi(tz)$, $1 > t > \max_{1 \leq j \leq l} |\lambda_j|$. Then $\varphi_t(\mathbb{D}) \subset\subset D$, $\varphi_t(0) = z$ and

$$\varphi_t(\xi) - a = \prod_{j=1}^{l} (\xi - \frac{\lambda_j}{t})^{k_j} \psi_t(z).$$

Since ψ_t is a bounded, it follows that for any $s = s_t < 1$, sufficiently close to 1, the mapping $\varphi_{s,t}$ defined by the formula

$$\varphi_{s,t}(\xi) = a + \prod_{j=1}^{l} \prod_{m=1}^{k_j} (\frac{\xi}{s^m} - \frac{\lambda_j}{t}) \psi_t(\xi)$$

belongs to the family $\mathcal{O}(\mathbb{D}, D)$, $\varphi_{s,t}(0) = z$, and the zeroes of the double product are pair-wise different. Thus

$$l_D^N(a,z) \leq \prod_{j=1}^{l} \prod_{m=1}^{k_j} s^m \frac{\lambda_j}{t}.$$

Letting $t \to 1$, $\epsilon \to 0$, $s \to 1$, we complete the proof. $\square$

Denote by $g_D(A, \cdot)$ *the pluricomplex Green function with pole at* $A \subset D$, i.e.

$$g_D(A,z) := \sup\{\exp(u(z))\},$$

where the supremum is taken over all $u : D \mapsto [-\infty, 0)$ such that $u(\cdot) - \log || \cdot - a||$ is bounded from above near any $a \in A$. Is it known that (cf. [3])

$$g_D(A,z) = \inf\{\prod_{\lambda \in \mathbb{D}} \chi_A(\varphi(\lambda))|\lambda|\} = \inf\{\prod_{\lambda \in \mathbb{D}} \chi_A(\varphi(\lambda)) \mathrm{ord}_\lambda \varphi |\lambda|\},$$

where the infimum is taken over all $\varphi \in \mathcal{O}(\mathbb{D}, D)$ with $\varphi(0) = z$. In particular, $g_D(A,z) \leq \tilde{l}_D(A,z)$ and the pluripotential Green function $g_D(a,z) := g_D(\{a\}, z)$ is equal to $\inf_{N \in \mathbb{N}} l_D^N(a,z)$ (cf. [3]). We shall see bellow that this infimum coincides with $l_D^\infty(a,z)$.

Note that the function $g_D(A, \cdot)$ is plurisubharmonic (cf. [3]), and the functions $l_D(A, \cdot)$, and l_D^N are upper semicontinuous.

Proposition 1. The sequence $(l_D^N(a,z))_{N\in\mathbb{N}}$ is decreasing and converges to $l_D^\infty(a,z)$.

Proof. Without loss of generality assume that $a\neq z$. To see the inequality
$$l_D^N(a,z)\geq l_D^{N+1}(a,z),$$
let $\varphi\in\mathcal{O}(\mathbb{D},D)$ be a competitor for $l_D^N(a,z)$ and $\lambda_1,\dots,\lambda_N\in\mathbb{D}$ be preimages of z. For $\alpha\in\mathbb{D}$, denote by
$$\Phi_\alpha(z):=\frac{\alpha-z}{1-\bar{\alpha}z}$$
the Möbius transformation. Observe that if $|\alpha|<1$ is close to 1, then one of the roots of the equation $z\Phi_\alpha(z)=\lambda_j$, say $\lambda_{j,1}$ is close to λ_j and the other one to 1. Then it is sufficient to take $\varphi(z\Phi_\alpha(z))\in\mathcal{O}(\mathbb{D},D)$ as a competitor for $l_D^{N+1}(a,z)$ and $\lambda_{1,1},\lambda_{2,1},\dots,\lambda_{N,1},\lambda_{N,2}\in\mathbb{D}$ as preimages of z.

To show that
$$\lim_{N\to\infty}l_D^N(a,z)=l_D^\infty(a,z),$$
observe first that
$$\liminf_{N\to\infty}l_D^N(a,z)\leq l_D^\infty(a,z)$$
by definitions. Thus, we have to prove that
$$l_D^N(a,z)\geq l_D^\infty(a,z)$$
for any $N\in\mathbb{N}$. Set $f(z)=z\exp(\frac{z-1}{z+1})$. We claim that for every $\lambda\in\mathbb{D}\setminus\{0\}$ there are infinitely many solutions of the equation $f(z)=\lambda$ from $\mathbb{D}$ and that the product of absolute values of these solutions coincides with $|\lambda|$. Indeed, this follows from the fact that the function $\Phi_\lambda\circ f$ has no zero radial limits and hence it is an infinite Blaschke product (cf. [3]). To complete the proof, similarly as above, we consider compositions of f with the competitors for $l_D^N(a,z)$. $\square$

3. Explicit formulas and extremal discs for $l_D(A,z)$ and $l_D^N(a,z)$

To obtain counterexamples to the product property of the Lempert functions, we shall need explicit formulas for $l_D(A,z)$ and $l_D^N(a,z)$, and descriptions of the extremal discs for these functions in the plane case.

A mapping $\varphi\in\mathcal{O}(\mathbb{D},D)$ is called an $l_D(A,z)$-*extremal disc* if $\varphi(0)=z$ and there exists a nonempty subset B of A with $l_D(A,z)=\prod_{a\in B}|\lambda_a|$, where $\varphi(\lambda_a)=a$ for any $a\in B$ (cf. [8, 9]). A mapping $\varphi\in\mathcal{O}(\mathbb{D},D)$ is said to be an $l_D^N(a,z)$-*extremal* if $\varphi(0)=z$ and $l_D^N(a,z)=\prod_{j=1}^M|\lambda_j|$, where $1\leq M\leq N$, $\varphi(\lambda_j)=a$, $j=1,\dots,M$, and we allow some of the

λ_j's to be equal but they cannot be counted more than the multiplicity $\operatorname{ord}_{\lambda_j}\varphi$ of φ at λ_j (compare with the definition of $\hat{l}_D^N$).

It is an easy observation that if D is taut (i.e, if the family $\mathcal{O}(\mathbb{D}, D)$ is normal), $\emptyset \neq A \subset D$ is finite, $a \in D$ and $N \in \mathbb{N}$, then there are $l_D(A, z)$-extremal discs and $l_D^N(a, z)$-extremal discs. Moreover, in this case the functions $l_D(A, \cdot)$ and $l_D^N(a, \cdot)$ are continuous.

Recall that a plane domain D is taut if and only if its boundary contains more than one point (cf. [4]). On the other hand, if the boundary of a plane domain D contains at most one point, then $l_D(A, \cdot) \equiv 0$ and $l_D^N(a, \cdot) \equiv 0$.

An application of the Schwarz-Pick lemma gives us the following explicit formulas in the case of the unit disc:

$$l_{\mathbb{D}}(A, z) = \tilde{l}_{\mathbb{D}}(A, z) = \prod_{a \in A} \left|\Phi_a(z)\right|, \; l_{\mathbb{D}}^N(a, z) = \left|\Phi_a(z)\right|.$$

Moreover, if $z \notin A$, then the $l_{\mathbb{D}}(A, z)$-extremal discs are the automorphisms of $\mathbb{D}$, sending 0 to z. If $z \neq a$, $n \in \mathbb{N}$, then one may easily see that the $l_{\mathbb{D}}^N(a, z)$-extremal discs are the Blaschke products of degree less than or equal to N, which map 0 into z.

Now, we are going to deal with the non-simply connected plane domains whose boundaries contain more than one point.

Proposition 2. *Let D be a non-simply connected plane domain whose boundary contains more than one point, $a, z \in D$ and $N \in \mathbb{N}^*$. Let $\pi \in \mathcal{O}(\mathbb{D}, D)$ be a cover map with $\pi(0) = z$. Assume that $\pi^{-1}(a) = \{\eta_1, \eta_2, \dots\}$ and $|\eta_1| \leq |\eta_2| \leq \dots$. Then*

$$l_D^N(a, z) = \prod_{j=1}^{N} |\eta_j|. \tag{1}$$

In particular, $l_D^N(a, z) < l_D^K(a, z)$ for $z \neq a$ and $K < N \leq \infty$.

If $A \subset D$, then

$$l_D(A, z) = \prod_{a \in A} \min\{|\eta| : \pi(\eta) = a\}.$$

Moreover, if $z \neq a$ and $N \in \mathbb{N}$ (respectively, $l_D(A, z) > 0$), then the $l_D^N(a, z)$-extremal discs (respectively, $l_D(A, z)$-extremal discs) are the functions of the form $\pi \circ r$, where r is a rotation.

Proof. We shall only prove the statements for $l_D^N(a, z)$, since the proof for $l_D(A, z)$ is similar.

Without loss of generality we may assume that $a \neq z$. Let $\varphi \in \mathcal{O}(\mathbb{D}, D)$ be an $l_D^N(a, z)$-extremal disc. Note that there exists $r \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ with $r(0) = 0$ and $\varphi = \pi \circ r$. Choose sequences $(\lambda_{j,k}) \subset \mathbb{D}$, $(\nu_{j,k}) \subset \mathbb{N}$, where $l_j \geq 1$, $j = 1, \dots, M$, $k = 1, \dots, l_j$ and $(\lambda_j) \subset \mathbb{D}$,

$j = 1, \dots, M$ such that all $\lambda_{j,k}$ (and all λ_j) are pairwise different $\sum_{j=1}^{M}\sum_{k=1}^{l_j}\nu_{j,k} \le N$, $\nu_{j,k} \le \operatorname{ord}_{\lambda_{j,k}}\varphi$, $r(\lambda_{j,k}) = \lambda_j$ and

$$\prod_{j=1}^{M}\prod_{k=1}^{l_j}|\lambda_{j,k}|^{\nu_{j,k}} = l_D^N(a,z).$$

Certainly, $M \le N$. Note that $\operatorname{ord}_{\lambda_{j,k}}\varphi = \operatorname{ord}_{\lambda_{j,k}} r$. Then it easily follows from the Schwarz Lemma that

$$\prod_{k=1}^{l_j}|\lambda_{j,k}|^{\nu_{j,k}} \ge |\lambda_j|,\ j = 1, \dots, M.$$

Therefore,

$$\prod_{j=1}^{M}|\lambda_j| \le \prod_{j=1}^{M}\prod_{k=1}^{l_j}|\lambda_{j,k}|^{\nu_{j,k}} \le \prod_{j=1}^{N}|\mu_j|.$$

Since $\pi(\lambda_j) = z$, $j = 1, \dots, M$, we easily get from the way we chose μ_j that $M = N$ and, up to a permutation of the sequence (λ_j) we also have $|\lambda_j| = |\mu_j|$, $j = 1, \dots, N$ and the inequalities above become equalities, which in view of the Schwarz Lemma implies that $l_j = \nu_{j,1} = 1$, $j = 1, \dots, N$ and finally r is a rotation.

□

Recall now that if the boundary of a plane domain is a polar set, then the usual Green function vanishes identically (see e.g. [4]). Otherwise, we have the following description of the l_D^∞-extremal discs (see [7]), which completes the picture in the plane case.

Proposition 3. *Let D be an arbitrary plane domain whose boundary is a non-polar set, $z \in D$, and let $\pi \in \mathcal{O}(\mathbb{D}, D)$ be a cover map with $\pi(0) = z$. If $z \ne a$, then the $l_D^\infty(a,z)$-extremal discs exist and they have the form $\pi \circ B$, where $B \in \mathcal{O}(\mathbb{D}, D)$, $B(0) = 0$ and $\Phi_\eta \circ B$ is a Blaschke product for any $\eta \in \pi^{-1}(a)$.*

4. Product property of the Lempert function

It is known that the Green function has the product property (cf. [3]). In this paragraph we shall prove a result describing when the product property of the Lempert function holds if one the pole sets is singleton. It is a slight generalization of Theorem 2.1 in [2]. As we already mentioned, the main point in the proof will be the following lemma whose proof (of a less general version) in [2] (see Lemma 2.3 there) seems to be false.

Lemma 4. *Let $N \in \mathbb{N}^*$, $\mu_1, \mu_2, \dots \in \mathbb{D}$, $p = \prod_{j=1}^N |\mu_j|$, and $q \in (p, 1)$. Then there exist $f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ and $\eta_1, \eta_2, \dots \in \mathbb{D}$ such that $\prod_{j=1}^N |\eta_j| = q$, $f(0) = 0$ and $f(\eta_j) = \mu_j$ for any j.*

Proof. The proof is similar to that of Proposition 1.

We may assume that $\mu_j \neq 0$ for any j. Otherwise we replace the numbers μ_j by their non-zero preimages under the mapping $z\Phi_\alpha(z) \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ for $\alpha \in \mathbb{D}$ sufficiently close to 1. We shall consider two cases.

Let first $p \neq 0$. We shall choose the desired function of the form $f_a(z) = z\Phi_a(z),\ a \in [0, 1)$. Note that the equation $f_a(z) = \mu_j$ has exactly two (counted with multiplicity) roots $z_j(a), w_j(a)$, and they both belong to $\mathbb{D}$. Assuming $|z_j(a)| \leq |w_j(a)|$, we have $|z_j(a)| \leq \sqrt{|\mu_j|} \leq |w_j(a)|$. Moreover, $|z_j(a)|$ and $|w_j(a)|$ depend continuously on a (to see this, use, for example, the formula for the solution of the equation $f_a(z) = \mu_j$). Note also that $|z_j(0)| = |w_j(0)| = \sqrt{|\mu_j|}$ and

$$\lim_{a\to 1} |z_j(a)| = |\mu_j|, \lim_{a\to 1} |w_j(a)| = 1.$$

Set

$$g(a) = \prod_{j=1}^N |z_j(a)|,\ h(a) = \prod_{j=1}^N |w_j(a)|,\ a \in [0, 1).$$

We claim that the functions g and h are continuous and if $a \to 1$, then $g(a) \to p$ and $h(a) \to 1$. We also have the equality $g(0) = h(0) = \sqrt{p}$. The only problem with these properties is the continuity of functions h and g in the case $N = \infty$, so assume that $N = \infty$. To prove the continuity, we easily see that both functions are upper semicontinuous. On the other hand, their lower semicontinuity follows by the inequalities

$$g(a) \geq \prod_{j=1}^M |z_j(a)| \prod_{j=M+1}^\infty |\mu_j|,\ h(a) \geq \prod_{j=1}^M |w_j(a)| \prod_{j=M+1}^\infty |\mu_j|,\ M \in \mathbb{N},$$

and the continuity of the first products.

So, if $q \leq \sqrt{p}$, then there exists an $a \in [0, 1)$ with $\prod_{j=1}^N |z_j(a)| = q$; otherwise, we find an $a \in [0, 1)$ with $\prod_{j=1}^N |w_j(a)| = q$.

We shall now consider the case $p = 0$. Having in mind the case proved above, it is sufficient to show that there exist $f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ and points $\eta_1, \eta_2, \dots \in \mathbb{D}$ such that $\displaystyle\prod_{j=1}^\infty |\eta_j| \in (0, q)$, $f(0) = 0$ and $f(\eta_j) = \mu_j$, for

any j. Fix k with $\prod_{j=1}^{k}|\eta_j|<q^2$, choose ε such that

$$\max_{|z|\le\sqrt{|\mu_j|}}|e^{\varepsilon\frac{z-1}{z+1}}-1|<1-\sqrt{|\mu_j|}$$

for $1\le j\le k$, and set $f=z\exp(\varepsilon\frac{z-1}{z+1})$. It follows by the Rouché theorem that the functions $z-\mu_j$ and $f-\mu_j$ have the same numbers of zeroes inside the disc $\{z\in\mathbb{C}:|z|<\sqrt{|\mu_j|}\}$. Hence for any $j\le k$ there is a unique η_j from this disc such that $f(\eta_j)=\mu_j$. On the other hand, similarly as in the proof of Proposition 1, the function $\Phi_{\mu_j}\circ f$ is an infinite Blaschke product. Therefore, for any $j>k$, we may choose η_j with $|\eta_j|>2^{-2^{-j}}$ and $f(\eta_j)=\mu_j$. Thus $\prod_{j=1}^{\infty}|\eta_j|$ is a non-zero product, smaller than q. $\square$

Now we are going to the main result in this section.

Theorem 5. *Let D and G be domains in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, and let $z\in D$, $w,b\in G$. Then for any nonempty at most countable $A\subset D$ the following inequalities hold:*

$$\max\{l_D(A,z),l_G^{\#A}(b,w)\}\le l_{D\times G}(A\times\{b\},(z,w))\le \max\{l_D(A,z),l_G(b,w)\}.$$

Moreover, for given $N\in\mathbb{N}^*$ the equality

$$l_{D\times G}(A\times\{b\},(z,w))=\max\{l_D(A,z),l_G(b,w)\}$$

holds for any $A\subset D$ with N elements if and only if $l_G(b,w)=l_G^N(b,w)$.

Proof. The proof is similar to that of Theorem 2.1 in [2]. The left hand-side inequality follows by the definitions. To prove the other one, let $\alpha<1$ be such that $\alpha>\max\{l_D(A,z),l_G(b,w)\}$. If $A=\{a_j\}_{j=1}^N$, then there exist $\varphi\in\mathcal{O}(\mathbb{D},D)$, $\lambda_j\in\mathbb{D}$, $\psi\in\mathcal{O}(\mathbb{D},G)$ and $\zeta\in\mathbb{D}$ such that $\varphi(0)=z$, $\varphi(\lambda_j)=a_j$, $\psi(0)=w$, $\psi(\zeta)=b$, and $\max\{\prod_{j=1}^N|\lambda_j|,\ |\zeta|\}<\alpha$. By Lemma 4 we may find $f\in\mathcal{O}(\mathbb{D},\mathbb{D})$ and $\eta_1,\eta_2,\dots\in\mathbb{D}$ such that $\prod_{j=1}^N|\eta_j|=\alpha$, $f(0)=0$ and $f(\eta_j)=\lambda_j$. Set

$$B=\prod_{j=1}^{N}\frac{\bar\eta_j}{|\eta_j|}\Phi_{\eta_j},\ \ \xi=(\varphi\circ f,\psi(\frac{\zeta}{\alpha}\Phi_\alpha\circ B).$$

Then $\xi\in\mathcal{O}(\mathbb{D},D\times G)$, $\xi(0)=(z,w)$ and $\xi(\eta_j)=(a_j,b)$, which implies that

$$l_{D\times G}(A\times\{b\},(z,w))\le\alpha.$$

Hence

$$l_{D\times G}(A\times\{b\},(z,w))\le\max\{l_D(A,z),l_G(b,w)\}$$

It remains to show that if

$$l_{D\times G}(A\times\{b\},(z,w))=\max\{l_D(A,z),l_G(b,w)\}$$

for any $A\subset D$ with N elements, then $l_G(b,w)\le l_G^N(b,w)$, because the opposite inequality always holds. We know that for any $\varepsilon>0$ there exist $\varphi\in\mathcal{O}(\mathbb{D},G)$ and pair-wise distinct points $\eta_1,\eta_2,\dots\in\mathbb{D}$ such that $\varphi(0)=w$, $\varphi(\eta_j)=b,\ j=1,2,\dots$ and

$$\prod_{j=1}^{\infty}|\eta_j|<g_G(b,w)+\varepsilon.$$

Note that we may choose $\psi\in\mathcal{O}(\mathbb{D},D)$ with $\psi(0)=w$ and $\psi(\eta_j)\ne\psi(\eta_k)$, if $j\ne k$. Set $A=\{\psi(\eta_j)\}_{j=1}^N$. Since $(\psi,\varphi)\in\mathcal{O}(\mathbb{D},D\times G)$ is a competitor for $l_{D\times G}(A\times\{b\},(z,w))$, we conclude that

$$l_G(b,w)\le l_{D\times G}(A\times\{b\},(z,w))\le l_G^N(b,w)+\varepsilon.\quad\square$$

Corollary 6. (see [2]) *Let D and G be domains in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, and let $z\in D$, $w,b\in G$. Then the equality*

$$l_{D\times G}(A\times\{b\},(z,w))=\max\{l_D(A,z),l_G(b,w)\}$$

holds for any nonempty at most countable $A\subset D$ if and only if $l_G(b,w)=g_G(b,w)$.

Note that by the Lempert theorem (cf. [4]) the last equality holds for any convex domains. It is also true for the symmetrized bidisc which is not biholomorphic to a convex domain (see [1], see also [**?**]).

5. Counterexamples to the product property of the Lempert function

Let G be a plane domain and let D be a domain in $\mathbb{C}^n$. Theorem 5 and the explicit formula for l_G^N (Proposition 2) show that the product property for $l_{D\times G}(A\times\{b\},(z,w))$, $b\ne w$, holds if and only if either G is simply connected or its complement is a singleton.

In this paragraph we shall see that the product property for the Lempert function of the bidisc is a seldom phenomenon if each of the pole sets has more than one element.

We also show that the left-hand side inequality in Theorem 5 is not a good candidate for a modified product property; namely, this inequality is strict, in general, for non-simply connected domains whose boundaries contain more than one point.

Since the Green function has the product property, it does not exceed the Lempert function and both functions coincide on the unit disc, it

follows that

$$\max\{l_{\mathbb{D}}(A,z), l_{\mathbb{D}}(B,w)\} \le l_{\mathbb{D}^2}(A\times B,(z,w)). \tag{2}$$

On the other hand, we have

Theorem 7. *Let A and B be two-point subsets of $\mathbb{D}$, such that $0\notin A,B$ and $l_{\mathbb{D}}(A,0)=l_{\mathbb{D}}(B,0)$. Then*

$$l_{\mathbb{D}}(A,0)=l_{\mathbb{D}^2}(A\times B,(0,0))$$

if and only if there is a rotation sending A to B.

In addition, if $B=e^{i\theta}A$, $\theta\in\mathbb{R}$, then the $l_{\mathbb{D}^2}(A\times B,(0,0))$-extremal discs are of the form $(r,e^{i\theta}r)$, where r is a rotation.

Proof. Let $A=\{a_1,a_2\}$, $B=\{b_1,b_2\}$, and let $\psi=(\psi_1,\psi_2)$ be an $l_{\mathbb{D}^2}(A\times B,(0,0))$-extremal disc. Then there are a set $J\subset\{1,2\}\times\{1,2\}$ and numbers $z_{k,l}\in\mathbb{D}$, $(k,l)\in J$, such that

$$\psi(z_{k,l})=(a_k,b_l) \text{ and } \prod_{(k,l)\in J}|z_{k,l}|=l_{\mathbb{D}^2}(A\times B,(0,0)).$$

Suppose now that $l_{\mathbb{D}}(A,0)=l_{\mathbb{D}^2}(A\times B,(0,0))$. Since

$$l_{\mathbb{D}}(A,0)=l_{\mathbb{D}}(a_1,0)\cdot l_{\mathbb{D}}(a_2,0),$$

it follows that ψ_1 is an $l^2_{\mathbb{D}}(a_j,0)$-extremal disc, $j=1,2$ Analogously, ψ_2 is an $l^2_{\mathbb{D}}(b_j,0)$-extremal disc, $j=1,2$. In particular, $\#J\ge 2$, and ψ_1,ψ_2 are rotations or Blaschke products of degree two.

If $\#J=3$, then ψ_1 or ψ_2 must be simultaneously a rotation and a Blaschke product of degree two, which is a contradiction.

Let $\#J=4$. Then we may assume that

$$\psi_1(z)=z\Phi_\alpha(z),\ \psi_2(z)=e^{it}z\Phi_\beta$$

for some $\alpha,\beta\in\mathbb{D},t\in\mathbb{R}$. Therefore,

$$z_{1,1}\Phi_\alpha(z_{1,1})=z_{1,2}\Phi_\alpha(z_{1,2}),\ z_{2,1}\Phi_\alpha(z_{2,1})=z_{2,2}\Phi_\alpha(z_{2,2}),$$

$$z_{1,1}\Phi_\beta(z_{1,1})=z_{2,1}\Phi_\beta(z_{2,1}),\ z_{1,2}\Phi_\beta(z_{1,2})=z_{2,2}\Phi_\beta(z_{2,2}).$$

It follows that that

$$z_{1,1}=\Phi_\alpha(z_{1,2})=\Phi_\beta(z_{2,1}),\ z_{1,2}=\Phi_\beta(z_{2,2}),\ z_{2,1}=\Phi_\alpha(z_{2,2}).$$

Hence $z_{1,1}=\Phi_\alpha\circ\Phi_\beta(z_{2,2})=\Phi_\beta\circ\Phi_\alpha(z_{2,2})$. Then a straightforward calculation leads to the equality

$$(2-\alpha\bar\beta-\bar\alpha\beta)(z_{2,2}^2(\bar\alpha-\bar\beta)+z_{2,2}(\alpha\bar\beta-\bar\alpha\beta)+\beta-\alpha)=0.$$

It is easy to see that if $\alpha\ne\beta$, then both roots of the equation

$$z^2(\bar\alpha-\bar\beta)+z(\alpha\bar\beta-\bar\alpha\beta)=\alpha-\beta$$

belong to $\partial\mathbb{D}$. Thus, $\alpha=\beta$, $z_{1,2}=z_{2,1}$, $z_{1,1}=z_{2,2}$, a contradiction. $\square$

It remains to consider the case $\#J = 2$. Then either $(1,2),(2,1) \notin J$, or $(1,1),(2,2) \notin J$. It follows that ψ_1 and ψ_2 must be rotations, say $\psi_1(z) = e^{i\theta_1}z$, $\psi_2(z) = e^{i\theta_2}z$, $\theta_1, \theta_2 \in \mathbb{R}$, and hence $B = e^{i\theta}A$, where $\theta = \theta_1 - \theta_2$.

Conversely, it is clear that if $B = e^{i\theta}A$ and r is a rotation, then the mapping $(r, e^{i\theta}r) \in \mathcal{O}(\mathbb{D}, \mathbb{D}^2)$ is a competitor for $l_{\mathbb{D}^2}(A\times B,(0,0))$. This implies that

$$l_{\mathbb{D}}(A,0) \geq \ l_{\mathbb{D}^2}(A \times B,(0,0)).$$

Now the inequality (2) completes the proof. □

A consequence of Theorem 7 is the following

Corollary 8. *Let A, B be two-point subsets of $\mathbb{D}$ and $z \in \mathbb{D}\setminus A$. Then there exist uncountable many $w \in \mathbb{D}$ for which*

$$l_{\mathbb{D}}(A,z) = l_{\mathbb{D}}(B,w) < l_{\mathbb{D}^2}(A\times B,(z,w)).$$

Proof. It suffices to note that there exist uncountable many w's with $l_{\mathbb{D}}(A,z) = l_{\mathbb{D}}(B,w)$, but at most two w's for which there is an automorphism of $\mathbb{D}$, sending z to w and A to B. □

We do not know whether Theorem 7 still holds for sets with equal numbers of elements, greater than 1. However, this theorem and the next proposition provide for given $(z,w) \in \mathbb{D}^2$ a large class of counterexamples to the product property of $l_{\mathbb{D}^2}(A\times B,(z,w))$ for pole sets A and B with arbitrary numbers of elements, greater than 1.

Proposition 9. *Let D and G be domains in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Let $z \in D,\ w \in G,\ A \subset D,\ B \subset G$ and $q \in (0,1)$ be such that*

$$\max\{l_D(A,z), l_G(B,w)\} = q l_{D\times G}(A\times B,(z,w)) > 0.$$

Then

$$\max\{l_D(A\cup A_1,z), l_G(B\cup B_1,w)\} < l_{D\times G}((A\cup A_1)\times(B\cup B_1),(z,w))$$

for any $A_1 \subset D,\ B_1 \subset G$ with $A\cap A_1 = B\cap B_1 = \emptyset$ and

$$g_D(A_1,z)g_G(B_1,w) > q. \tag{3}$$

Proof. It is easy to see that

$$l_{D\times G}((A\cup A_1)\times(B\cup B_1),(z,w)) \geq$$
$$l_{D\times G}(A\times B,(z,w)) l_{D\times G}(A\times B_1,(z,w)) l_{D\times G}(A_1\times(B\cup B_1),(z,w))$$
$$\geq l_{D\times G}(A\times B,(z,w)) g_{D\times G}(A\times B_1,(z,w)) g_{D\times G}(A_1\times(B\cup B_1),(z,w))$$
$$\geq l_{D\times G}(A\times B,(z,w)) g_G(B_1,w) g_D(A_1,z)$$
$$> \max\{l_D(A,z), l_G(B,w)\} \geq \max\{l_D(A\cup A_1,z), l_G(B\cup B_1,w)\}. \quad \square$$

Remark. Recall that if the boundary of a planar domain D is a non-polar set, then there exists a polar set $F \subset \partial D$ such that $\lim_{a \to a_0} g_D(a, z) = 1$ for any $a_0 \in (\partial D) \setminus F$ and any $z \in D$. Since

$$g_D(A, z) = \prod_{a \in A} g_D(a, z),$$

it follows that for a given $q \in (0, 1)$ and $N \in \mathbb{N}^*$ there is a set A with N elements and with $\operatorname{dist}(A, a_0) < 1 - q$, and $g_D(A, z) > q$. So, we may provide the inequality (3) for any planar domains whose boundaries are non-polar.

Now we shall prove two results showing that the left-hand side inequality in Theorem 5 is also strict for general plane domains.

Proposition 10. *Let D and G be plane domains whose boundaries contain more than one point, $w, b \in G$, $w \neq b$, $z \in D$. Assume that G is non-simply connected. Then there exists a countable set $A = \{a_1, a_2, \dots\}$ of points in D such that if $A_N = \{a_1, a_2, \dots, a_N\}$, $N \in \mathbb{N} \setminus \{1\}$, then*

$$l_D(A_N, z) = l_G^N(b, w) < l_{D \times G}(A_N \times \{b\}, (z, w)).$$

Moreover, if the boundary of G is a non-polar set, then

$$l_D(A, z) = g_G(b, w) < l_{D \times G}(A \times \{b\}, (z, w)).$$

Proof. Since $l_D(\cdot, z)$ is a continuous function, $l_D(z, z) = 0$ and $\lim_{a \to \partial D} l_D(a, z) = 1$ (which follows by the explicit formula for $l_D(a, z)$), we may find a_1 with $l_D(a_1, z) = l_G(b, w) > 0$. Using similar argument, we obtain a sequence of points $a_1, a_2, \dots \in D$ such that

$$l_D(A_N, z) = l_G^N(b, w) > 0.$$

Moreover, each of these points can be chosen in uncountable many ways. Thus, if $\pi \in \mathcal{O}(\mathbb{D}, G)$ and $\tau \in \mathcal{O}(\mathbb{D}, D)$ are cover maps with $\pi(0) = w$ and $\tau(0) = z$, then we may assume that

$$\frac{\xi_1}{\xi_2} \neq \frac{\eta}{\zeta} \text{ for any } \xi_1 \in \tau^{-1}(a_1),\ \xi_2 \in \tau^{-1}(a_2),\ \eta, \zeta \in \pi^{-1}(b). \tag{4}$$

Suppose now that for some $N \in \mathbb{N} \setminus \{1\}$ we have

$$l_D(A_N, z) = l_{D \times G}(A_N \times \{b\}, (z, w)).$$

Since $D \times G$ is a taut domain, there exists an $\tilde{l}_{D \times G}(A_N \times \{b\}, (z, w))$-extremal disc (φ, ψ). Then φ and ψ must be $\tilde{l}_D(A_N, z)$-extremal disc and $l_D^N(b, w)$-extremal disc, respectively. By Proposition 2, we may assume that $\psi = \pi$ and $\varphi = \tau \circ e^{i\theta}$ for some real θ. In particular,

there are $\eta_1 \in \pi^{-1}(b) \cap e^{-i\theta}\tau^{-1}(a_1)$ and $\eta_2 \in \pi^{-1}(b) \cap e^{-i\theta}\tau^{-1}(a_2)$. A contradiction with (4).

We are going to the second part of the proposition. First, we shall show that there exists an $l_{D\times G}(A \times \{b\}, (z, w))$-extremal disc. Let ξ_N, $N \in \mathbb{N}$, be an $l_{D\times G}(A_N \times \{b\}, (z, w))$-extremal disc. Then there exist sets $J \subset \{1, 2, \dots, N\}$ and $(\lambda_{j,N})_{j\in J} \subset \mathbb{D}$ such that $\xi_N(0) = (z, w)$, $\xi_N(\lambda_{j,N}) = (a_j, b)$ for any $j \in J$, and

$$l_{D\times G}(A_N \times \{b\}, (z, w)) = \prod_{j\in J} |\lambda_{j,N}|.$$

Put $\lambda_{j,N} = 1$ for $j \notin J$. Passing to subsequences and applying the standard diagonal process, we may assume that $\xi_N \to \xi \in \mathcal{O}(\mathbb{D}, D\times G)$ uniformly on compact subsets of $\mathbb{D}$, $\lim_{N\to\infty} \lambda_{j,N} = \lambda_j \in \overline{\mathbb{D}}$ for any j and $\xi(0) = (z, w)$, $\xi(\lambda_j) = (a_j, b)$. To prove that ξ is an $l_{D\times G}(A \times \{b\}, (z, w))$-extremal disc, suppose the contrary. It follows that

$$\prod_{j=1}^{\infty} |\lambda_j| \geq q l_{D\times G}(A \times \{b\}, (z, w)),$$

where $q > 1$. Then for any k there is n_k such that

$$\prod_{j=1}^{k} |\lambda_{j,N}| \geq q l_{D\times G}(A \times \{b\}, (z, w))$$

if $N \geq n_k$. Since

$$\prod_{j=k+1}^{\infty} |\lambda_{j,N}| \geq l_D(A \setminus A_k, z),$$

we obtain that

$$l_{D\times G}(A_N \times \{b\}, (z, w)) \geq q l_D(A \setminus A_k, z) l_{D\times G}(A \times \{b\}, (z, w)). \quad (5)$$

Note that the assumption that the boundary of G is a non-polar set implies

$$l_D(A, z) = g_G(b, w) > 0.$$

Then (among others we use the fact that the sequence (a_k) has no accumulation point in D)

$$\lim_{N\to\infty} l_{D\times G}(A_N \times \{b\}, (z, w)) = l_{D\times G}(A \times \{b\}, (z, w)) > 0,$$

$$\lim_{k\to\infty} l_D(A \setminus A_k, z) = 1.$$

It follows by (5) that $1 \geq q$, a contradiction.

Thus, $\xi = (\varphi, \psi)$ is an $l_{D\times G}(A \times \{b\}, (z, w))$-extremal disc. Suppose now that

$$l_D(A, z) = g_G(b, w) = l_{D\times G}(A \times \{b\}, (z, w)).$$

Then φ and ψ must be a $l_D(A,z)$-extremal disc and an $l_G^\infty(b,w)$-extremal disc, respectively. By Propositions 2 and 3, we may assume that $\varphi=\tau$ and $\psi=\pi\circ B$, where $B\in\mathcal{O}(\mathbb{D},\mathbb{D})$, $B(0)=0$ and $\Phi_\eta\circ B$ is a Blaschke product for any $\eta\in\pi^{-1}(b)$. Moreover, $J=\mathbb{N}$, $\lambda_j\in\tau^{-1}(a_j)$, $|\lambda_j|=l_D(a_j,z)$ and $B(\lambda_j)\in\pi^{-1}(b)$ for any j. On the other hand,

$$|\lambda_1|\geq|B(\lambda_1)|\geq l_G(b,w)=l_D(a_1,z)=|\lambda_1|,$$

which shows that B is a rotation. Then, as above, we get a contradiction with (4), which completes the proof. □

This proof allows us to obtain also the following

Proposition 11. *Let D and G be plane domains whose boundaries contain more than one point, $w,b\in G$, $w\neq b$, $z\in D$. Assume that G is non-simply connected. Then there exists a countable subset A of D such that*

$$\max\{l_D(A,z),g_G(b,w)\}<l_{D\times G}(A\times\{b\},(z,w)).$$

Proof. Choose the points a_1 and a_2 in D as in the proof of Proposition 10 and set $A_2=\{a_1,a_2\}$. Let $q\in(0,1)$ be such that

$$l_D(A_2,z)=l_G^2(b,w)=q\,l_{D\times G}(A_2\times\{b\},(z,w)).$$

Now it is enough to note that for any countable subset $B\subset D$ with $l_D(B,z)>q$ we have

$$l_{D\times G}((A_2\cup B)\times\{b\},(z,w))\geq l_{D\times G}(A_2\times\{b\},(z,w))l_D(B,z)>$$

$$l_D(A_2,z)=l_G^2(b,w)>\max\{l_D(A_2\cup B,z),g_G(b,w)\}.\quad\square$$

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References

[1] C. Costara, *The symmetrized bidisc as a counterexample to the converse of Lempert's theorem,* Bull. London Math. Soc., to appear.

[2] N. Q. Dieu, N. V. Trao, *Product property of certain extremal functions,* Complex Variables, **48(8)** (2003), 681–694.

[3] A. Edigarian, *Analytic discs method in complex analysis,* Diss. Math., **402** (2002), 1-56.

[4] M. Jarnicki, P. Pflug, *Invariant distances and metrics in complex analysis,* De Gruyter (1993).

[5] M. Jarnicki, P. Pflug, *Invariant distances and metrics in complex analysis - revisited,* Diss. Math., to appear.

[6] N. Nikolov, P. Pflug, *The multipole Lempert function is monotone under inclusion of sets,* Preprint (2004).
[7] N. Nikolov, W. Zwonek, *Some remarks on the Green function and the Azukawa pseudometric,* Monats. Math., **142(4)** (2004), 341–350.
[8] F. Wikström, *Non-linearity of the pluricomplex Green function,* Proc. Amer. Math. Soc., **129(4)** (2001), 1051–1056.
[9] F. Wikström, *Qualitative properties of biholomorphically functions with multiple poles,* Preprint (2004).

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